Quark-antiquark potential in the analytic approach to QCD

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Abstract

The quark-antiquark potential is constructed by making use of a new analytic running coupling in QCD. This running coupling arises under analytization of the renormalization group equation. The rising behavior of the quark-antiquark potential at large distances, which provides the quark confinement, is shown explicitly. At small distances the standard behavior of this potential originated in the QCD asymptotic freedom is revealed. The higher loop corrections and the scheme dependence of the approach are briefly discussed.

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I. INTRODUCTION

The description of quark dynamics inside the hadrons remains an actual problem of the elementary particle theory. The asymptotic freedom in the Quantum Chromodynamics (QCD) enables one to investigate the quarks interaction at small distances by making use of the standard perturbation theory. The quark dynamics at large distances (the confinement region) lies beyond such calculations. For this purpose other approaches are used: the phenomenological potential models [1], the string models [2], the bags models [3], the lattice calculations [4], the explicit account of nontrivial QCD vacuum structure [5], the variational perturbation theory [6].

Recently a new analytic approach to QCD has been proposed [7]. Its basic idea is to combine the renormalization group (RG) summation with the analyticity requirement. The essential merits of this approach are the following: absence of unphysical singularities at any loop level, stability in the infrared (IR) region, stability with respect to loop corrections, and extremely weak scheme dependence. The analytic approach has been applied successfully to such problems as the τ lepton decays, e^+e^--annihilation into hadrons, sum rules (see [7] and references therein).

In the work [8] the analytic approach has been employed to the solution of the RG equation. The analyticity requirement was imposed on the RG equation itself, before deriving its solution. Solving the RG equation, analytized in the above-mentioned way, one gets, at one-loop level, a new analytic running coupling [8], which possesses practically the same appealing features as the Shirkov-Solovtsov running coupling [7] does. An essential distinction, that will play a crucial role in the present paper, is the IR singularity of the new analytic running coupling at the point $q^2 = 0$.

In this paper we shall adhere to the model [5], [9] of obtaining the quark-antiquark ($q\bar{q}$) potential by the Fourier transformation of the running coupling. However, the perturbative running coupling $\alpha_{\mathrm{s}}(q^2)$ does not enable one to obtain the rising $q\bar{q}$ potential without invoking additional assumptions [9].

The objective of this paper is to construct the quark-antiquark potential by making use of the new analytic running coupling. This potential proves to be rising at large distances (i.e., providing the quark confinement) and, at the same time, it incorporates the asymptotic freedom at small distances. It is essential that for obtaining this potential *no any additional assumptions*, lying beyond the standard RG method in the Quantum Field Theory and the analyticity requirement, will be used.

The layout of the paper is as follows. In Sec. II the quark-antiquark potential, generated by the new analytic running coupling is derived by making use of the Fourier transformation. Further the asymptotic behavior of the potential at large and small distances is investigated. In Sec. III the higher loop corrections and the scheme dependence of the potential are discussed briefly. For practical purpose a simple approximate formula for the potential is proposed which interpolates its infrared and ultraviolet asymptotics. This formula is compared with the phenomenological Cornell potential. Proceeding from this an estimation of the QCD parameter Λ is obtained. In the Conclusion (Sec. IV) the obtained results are formulated in a compact way, and the further studies in this approach are outlined.

II. QUARK-ANTIQUARK POTENTIAL GENERATED BY THE NEW ANALYTIC RUNNING COUPLING

We proceed from the standard expression [5], [9] for the $q\bar{q}$ potential in terms of the running coupling $\alpha(q^2)$

$$V(r) = -\frac{16\pi}{3}\int_0^\infty \frac{\alpha(q^2)}{q^2}\frac{e^{i\mathbf{qr}}}{(2\pi)^3}\,d\mathbf{q}. \tag{1}$$

For the construction of the new interquark potential ${}^N V(r)$ we shall use the new analytic running coupling [8]

$${}^{\mathrm{N}}\alpha_{\mathrm{an}}(Q^2) = \frac{4\pi}{\beta_0}\frac{z-1}{z\ln z}, \qquad z = \frac{Q^2}{\Lambda^2}, \tag{2}$$

where $\beta_0 = 11 - 2\,n_f/3$ is the first coefficient of the β-function. Upon the integration over the angular variables and the substitution $q/\Lambda \to q$, $r\Lambda \to R$ in Eq. (1) one gets

$${}^N V(r) = -\frac{32}{3\beta_0}\Lambda \cdot \widetilde{V}(R), \quad R = \Lambda r, \tag{3}$$

where

$$\widetilde{V}(R) = \int_0^\infty \frac{q^2-1}{q^2\ln q^2}\frac{\sin(qR)}{qR}\,dq \tag{4}$$

is the dimensionless potential.

In order to perform the integration in Eq. (4) we consider the auxiliary function

$$I(n,R) = \lim_{a\to 0+}\int_0^\infty \frac{q^{n-1}}{\ln(a+q^2)}\sin(qR)\,dq. \tag{5}$$

Here the parameter a is introduced for shifting the origin of the cut along the imaginary axis Im q. It is obvious that

$$\widetilde{V}(R) = \frac{1}{R}\left[I(0,R) - I(-2,R)\right]. \tag{6}$$

For even n the integrand in Eq. (5) is an even function of q. Therefore

$$I(n,R) = \frac{1}{2}\,\mathrm{Im}\,\lim_{a\to 0+} J(n,a,R), \tag{7}$$

where

$$J(n,a,R) = \mathcal{P}\int_{-\infty}^{\infty} F(q)\,dq, \quad F(q) = \frac{q^{n-1}e^{iqR}}{\ln(a+q^2)}. \tag{8}$$

The sign $\mathcal{P}$ means the principal value of the integral.

The function $F(q)$ in Eq. (8) has the cuts $(-i\infty, -i\sqrt{a}]$, $[i\sqrt{a}, i\infty)$ and simple poles at the points $q = \mp\sqrt{1-a}$. Let us consider the integral of the function F along the contour Γ shown in Fig. 1. The function $F(q)$ has no singularities inside the contour Γ, therefore $\oint_\Gamma F\, dq = 0$. Contribution to this integral of the semicircle of infinitely large radius in upper half-plane (see Fig. 1) vanishes. Performing the integration along the two semicircles c_- and c_+ of the vanishing radius and along the cut C on the imaginary axis, we obtain

$$J(n,a,R) = i\pi\Bigg\{\frac{1}{2}\left[(\sqrt{1-a})^{n-2}e^{iR\sqrt{1-a}}(-\sqrt{1-a})^{n-2}e^{-iR\sqrt{1-a}}\right] + 2i^{n-2}\int\limits_{\sqrt{a}}^{\infty}\frac{x^{n-1}e^{-Rx}}{\ln^2(x^2-a)+\pi^2}\,dx\Bigg\}. \tag{9}$$

Hence, for even n the function $I(n,R)$ in Eq. (5) takes the form

$$I(n,R) = \pi\left[\frac{1}{2}\cos R - (-1)^{n/2}\mathcal{N}(R,n)\right], \tag{10}$$

where

$$\mathcal{N}(R,n) = \int\limits_0^\infty \frac{x^{n-1}e^{-Rx}}{\ln^2(x^2)+\pi^2}\,dx. \tag{11}$$

It is rather complicated to perform the integration in Eq. (11) explicitly. Therefore we address the study of the asymptotics. First of all, we would like to know whether the $q\bar{q}$ potential ${}^NV(r)$ in Eq. (3) provides the quark confinement. For the investigation of the potential behavior at large distances it is enough to consider the asymptotic of the function $\mathcal{N}(R,n)$ in Eq. (11) when $R \to \infty$. This function can be represented in the following way

$$\mathcal{N}(R,n) = (-1)^n\frac{\partial^n}{\partial R^n}\int\limits_0^\infty \frac{e^{-Rx}}{x\left[\ln^2(x^2)+\pi^2\right]}\,dx. \tag{12}$$

At large R the basic contribution into Eq. (12) gives the integration over the small x region. Let us transform $\mathcal{N}(R,n)$ identically:

$$\mathcal{N}(R,n) = \frac{\partial^n}{\partial R^n}\int\limits_0^\infty \frac{(-1)^n e^{-Rx}}{4x(\ln^2 x+\pi^2)}\left[1+\frac{3L}{1+L}\right]dx, \tag{13}$$

where $L = \pi^2/(4\ln^2 x)$. Neglecting the second term in the square brackets in Eq. (13), we use the formula (4.361.2) from Ref. [10]:

$$\int\limits_0^\infty \frac{e^{-\mu x}}{x(\ln^2 x+\pi^2)}dx = e^\mu - \nu(\mu), \quad \text{Re}\,\mu > 0, \tag{14}$$

where $\nu(\mu)$ is the so-called transcendental ν-function [11]:

$$\nu(\mu) = \int\limits_0^\infty \frac{\mu^t\,dt}{\Gamma(t+1)}. \tag{15}$$

Eventually, we obtain for $R \to \infty$

$$\mathcal{N}(R,n) \simeq \frac{(-1)^n}{4}\left[e^R - \nu(R) + \int\limits_0^{-n} \frac{R^t dt}{\Gamma(t+1)}\right]. \tag{16}$$

Taking into account Eqs. (6), (10), and (16) one can present the quark-antiquark potential (3) at large R in the following way:

$$^N V(r) \simeq \frac{8\pi}{3\beta_0}\frac{\Lambda}{R}\left[2\left(e^R - \nu(R)\right) + \int\limits_0^{2} \frac{R^t dt}{\Gamma(t+1)}\right]. \tag{17}$$

The behavior of the potential $^N V(r)$ at $r \to \infty$ is determined by the last term in Eq. (17).[1] Integration of this term by parts gives

$$\int\limits_0^{2} \frac{R^t dt}{\Gamma(t+1)} = \frac{R^2}{2\ln R}\left[1 + 2\sum_{k=1}^{\infty}\frac{f_k(2)}{\ln^k R}\right] - \frac{1}{\ln R}\left[1 + \sum_{j=1}^{\infty}\frac{f_j(0)}{\ln^j R}\right], \tag{18}$$

where

$$f_n(t) = \frac{d^n}{ds^n}\frac{(-1)^n}{\Gamma(s+1)}\bigg|_{s=t}. \tag{19}$$

In the limit $R \to \infty$ Eq. (18) takes the form

$$\int\limits_0^{2} \frac{R^t dt}{\Gamma(t+1)} = \frac{R^2}{2\ln R}. \tag{20}$$

Therefore the quark-antiquark potential $^N V(r)$ proves to be rising at large distances

$$^N V(r) \simeq \frac{8\pi}{3\beta_0}\Lambda \cdot \frac{1}{2}\frac{\Lambda r}{\ln(\Lambda r)}, \qquad r \to \infty. \tag{21}$$

Thus the new analytic running coupling $^{\mathrm{N}}\alpha_{\mathrm{an}}(q^2)$ (see Eq. (2)) leads to the rising quark-antiquark potential $^N V(r)$ which can, in principle, describe the quark confinement.

It is important to point out that the behavior of the potential $^N V(r)$ when $r \to 0$ has the standard form determined by the asymptotic freedom (e.g., see Ref. [9])

$$^N V(r) \simeq \frac{8\pi}{3\beta_0}\Lambda \cdot \frac{1}{\Lambda r \ln(\Lambda r)}, \qquad r \to 0. \tag{22}$$

Unfortunately, it is impossible to obtain the explicit dependence $^N V(r)$ for the whole region $0 < r < \infty$. A simple interpolating formula, which can be applied for the practical use, will be given in the next section.

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[1] It follows directly from the asymptotic of $\nu(R)$ (see Ref. [11]), and from a simple reasoning. Really, if $R > 0$ the term $f(R) = e^R - \nu(R)$ is nonnegative and $f'(R) \leq 0$. Hence, $f(R) \to \mathrm{const}$ when $R \to \infty$, and its contribution to $^N V(r)$ at large R is of $1/R$-order.

III. DISCUSSION

Let us discuss briefly the higher loop contribution. One can show that the singularity of i-loop analytic running coupling ${}^{\rm N}\alpha_{\rm an}^{\rm (i)}(q^2)$ at the point $q=0$ is of the universal type at any loop level. Therefore, when $q \to 0$ we have ${}^{\rm N}\alpha_{\rm an}^{\rm (i)}(q^2) \sim {}^{\rm N}\alpha_{\rm an}^{(1)}(q^2)\, C^i$, where C^i are constants. Taking into account that the maximal difference between ${}^{\rm N}\alpha_{\rm an}^{\rm (i)}(q^2)$ and ${}^{\rm N}\alpha_{\rm an}^{(1)}(q^2)$ is in the small q^2 region, we arrive at the following conclusion. The account of the higher loop corrections leads to changing the slope of the $q\bar{q}$ potential ${}^{N}V(r)$ when $r \to \infty$. This corresponds to a simple redefinition of the parameter Λ in Eq. (21) at the higher loop levels.

As far as the scheme dependence of this approach, we have to point out the following. It was shown in [8] that the solutions of the analytized RG equation at the higher loop level have extremely weak scheme dependence. In particular, the solutions of the two-loop RG equation with $\overline{\rm MS}$ and MS schemes, are practically coinciding. Hence, at the higher loop level (there is no scheme dependence at the one-loop level), the use of different subtraction schemes leads to the slight variation of the $q\bar{q}$ potential.

Thus, neither higher loop corrections, nor scheme dependence can affect qualitatively the result obtained in the previous section.

For the practical use of the new potential it is worth obtaining a simple explicit expression that approximates it sufficiently well. For this purpose one can use, for instance, the approximating function

$$U(r) = \frac{8\pi}{3\beta_0}\Lambda\left[\frac{1}{\ln R}\left(\frac{1}{R}+\frac{R}{2}\right)+\frac{1}{1-R}\left(\frac{3}{2}+Rf_1(2)\right)\right.$$
$$\left.+Rf_1(2)\left(\frac{1}{\ln^2 R}-\frac{1}{(R-1)^2}+\frac{11}{12}\right)\right], \tag{23}$$

which has no any unphysical singularities and possesses the asymptotics (21) and (22). This function is obtained by smooth sewing the asymptotics

$${}^{N}V(r) \simeq \frac{8\pi}{3\beta_0}\Lambda\cdot\left[\frac{R}{2\ln(R)}+\frac{Rf_1(2)}{\ln^2(R)}\right], \quad R\to\infty, \tag{24}$$
$${}^{N}V(r) \simeq \frac{8\pi}{3\beta_0}\Lambda\cdot\frac{1}{R\ln(R)}, \quad R\to 0, \quad R=\Lambda r. \tag{25}$$

The formula (24) keeps explicitly the second leading term of the expansion (18), $f_1(2) = 0.461$. Some terms have been introduced into Eq. (23) only for eliminating the singularity at the point $R=1$. It should be mentioned here that the next terms in the expansion (18) practically do not affect the shape of $U(r)$. Of course, the function (23) is not the unique interpolating function between asymptotics (24) and (25). Nevertheless, the comparison of $U(r)$ with the phenomenological potential

$${}^{C}V(r) = -\frac{4}{3}\frac{a}{r}+\sigma r+\text{const} \tag{26}$$

(the so-called Cornell potential [1]) shows their almost complete coincidence (see Fig. 2). The fit has been performed with the use of the least square method in the physical meaning region $0.1 \le r \le 1.0$ fm [5]. The varied parameter in Eq. (23) is Λ. The possibility of shifting the potential ${}^{C}V(r)$ in Eq. (26) by a constant was also used. A rough estimation of Λ in the course of this fitting gives $\Lambda \simeq 500$ MeV. This is in agreement with the values obtained earlier in the framework of the analytic approach to QCD [7].

IV. CONCLUSION

In the paper the quark-antiquark potential is constructed by making use of the new analytic running coupling in QCD. This running coupling arises under analytization of the renormalization group equation before its solving. The rising behavior of the quark-antiquark potential at large distances, which provides the quark confinement, is shown explicitly. The key property of the new analytic running coupling, leading to the confining potential, is its infrared singularity at the point $q^2 = 0$. At small distances, the standard behavior of the potential, originated in the QCD asymptotic freedom, is revealed. It is also demonstrated that neither higher loop corrections, nor scheme dependence can affect qualitatively the obtained result. The estimation of the parameter Λ in this approach gives a reasonable value, $\Lambda \simeq 500$ MeV.

In further studies it would undoubtedly be interesting to consider in this approach the dependence of the $q\bar{q}$ potential on the quark masses.

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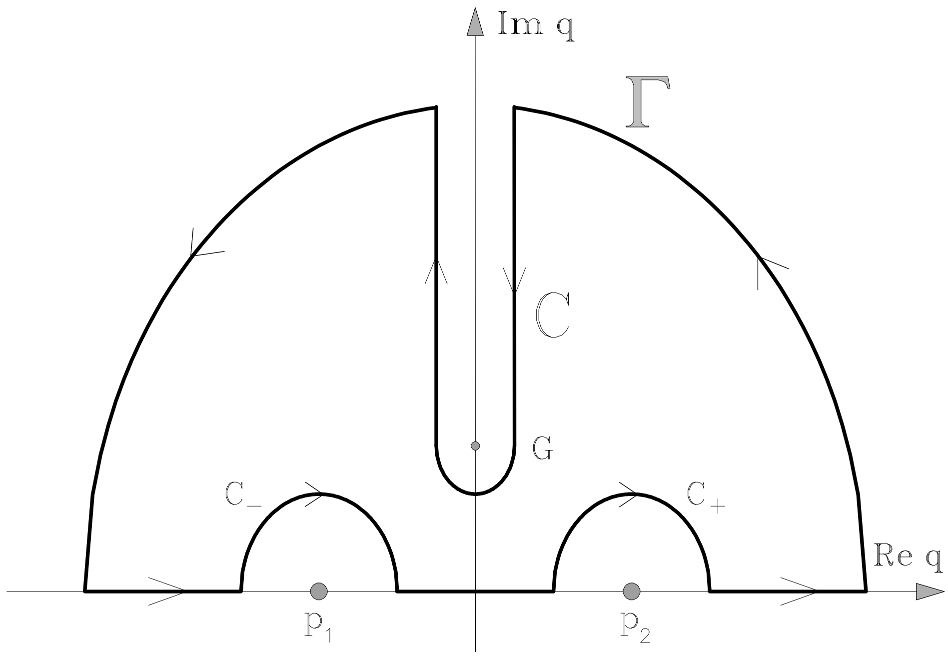

FIG. 1. The integration contour in the complex q-plane. The notations are: $p_1 = -\sqrt{1-a}$, $p_2 = \sqrt{1-a}$, G$= i\sqrt{a}$.

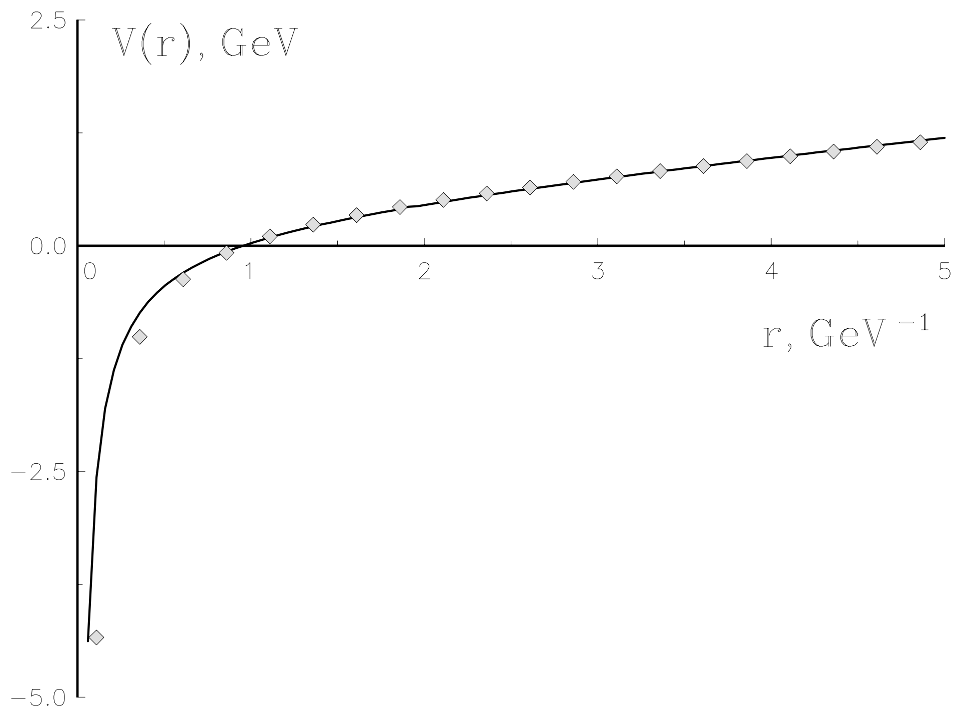

FIG. 2. Comparison of the potential $U(r)$ given by Eq. (23) (solid curve) with the phenomenological Cornell potential ($\diamond$), Eq. (26). The values of the parameters are: $a = 0.39$, $\sigma = 0.182$ GeV^2 [5], $\Lambda = 530\,$MeV, $n_f = 5$.